## 3.13

## Vectors in the Plane

A **two-dimensional vector v** is an ordered pair of real numbers, denoted in **component form** as $\langle a,b \rangle$. The numbers *a* and *b* are the **components** of the vector **v**.

The **standard representation** of the vector $\langle a,b \rangle$ is the arrow from the origin to the point (*a*, *b*). The **magnitude** of **v** is the length of the arrow, and the **direction** of **v** is the direction in which the arrow is pointing. The vector **0** = $\langle 0,0 \rangle$, called the **zero vector**, has zero length and no direction.

**Magnitude** (size) – single real number indicating its magnitude (temperature, distance, height, area, etc.)

**Magnitude and direction** –(force, velocity, acceleration) These are represented by ***directed line segments.***

If P and Q are points in the plane, then the direction from P to Q is indicated by an arrow. The **directed line segment** from P to Q is denoted by $\overrightarrow{PQ}$. **Initial point** is P and **terminal point** is Q .

Use the symbol $\left|\overrightarrow{PQ}\right|$ to represent the **length** or **magnitude** of the directed line segment $\overrightarrow{PQ}$. The direction of $\overrightarrow{PQ}$ is from P to Q. The

directed line segment $\overrightarrow{QP}$ has the same length as $\overrightarrow{PQ}$ but the opposite direction from Q to P.

Two directed line segments with the same length and direction are **equivalent.** Location of a vector doesn't matter, only its direction and magnitude matter.

The set of all directed line segments equivalent to the directed line segment $\overrightarrow{PQ}$ is the **vector v** = $\overrightarrow{PQ}$ and **v** is the vector represented by $\overrightarrow{PQ}$.

To denote vectors: use lowercase boldface letters. The length and direction of **v** = $\overrightarrow{PQ}$ is the same as the length and direction of $\overrightarrow{PQ}$. Two vectors are equal if their corresponding directed line segments are equivalent.

**<u>Head Minus Tail (HMT) Rule:</u>**

If an arrow has initial point $\left(x_{1,}y_1\right)$ and terminal point $\left(x_{2,}y_2\right)$, it represents the vector $\langle x_2-x_1, y_2-y_1\rangle$.

**Example:**

Show that the arrow from $R=(-4,2)$ to $S=(-1,6)$ is equivalent to the arrow from $P=(2,-1)$ to $Q=(5,3)$.

Using the HMT rule, $\overrightarrow{RS}$ is the vector $\langle -1-(-4), 6-2\rangle = \langle 3,4\rangle$, and $\overrightarrow{PQ}$ is the vector $\langle 5-2, 3-(-1)\rangle = \langle 3,4\rangle$.

## Component Form of a Vector

If **v** is a vector in the plane equal to the vector with initial point (0, 0) and terminal point ($v_1$, $v_2$), then the component form of **v** is $v = \langle v_1, v_2\rangle$ .

Numbers $v_1$, $v_2$ are the **components** of **v.** The vector $\langle v_1, v_2\rangle$ is called the **position vector** of the point ($v_1$, $v_2$)

The **magnitude** or **length** of the vector **v** = $\overrightarrow{PQ}$ determined by P($x_1$,$y_1$) and Q($x_2$,$y_2$) is

$$|v| = \sqrt{v_1^{\ 2} + v_2^{\ 2}} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Example:**

Find the magnitude of the vector **v** represented by $\overrightarrow{PQ}$, where $P=(-3,4)$ and $Q=(-5,2)$.

Solution: $|v|=\sqrt{(-5-(-3))^2+(2-4)^2}=2\sqrt{2}$

Or, using HMT rule **v** = $\langle -2,-2\rangle$, so $|v|=\sqrt{(-2)^2+(-2)^2}=2\sqrt{2}$

## <u>Vector Operations</u>

### Vector Addition and Scalar Multiplication:

Let **u** = $\langle u_1, u_2\rangle$ and **v**=$\langle v_1, v_2\rangle$ be vectors and *k* a real number (scalar). Then the **sum of the vectors u and v** is the vector **u** + **v** = $\langle u_1, u_2\rangle + \langle v_1, v_2\rangle = \langle u_1+v_1, u_2+v_2\rangle$.

The **product of the scalar *k* and the vector u** is

k**u =** k$\langle u_1, u_2\rangle$ = $\langle ku_1, ku_2\rangle$.

**Example:**

Let **u** =$\langle -1,3\rangle$ and **v** =$\langle 4,7\rangle$. Find the component form of the following vectors.

a) **u** + **v** b) 3**u** c) 2**u** + (-1)**v**

a) **u** + **v** = $\langle -1,3\rangle + \langle 4,7\rangle = \langle -1+4, 3+7\rangle$

b) 3**u** = $3\langle -1,3\rangle$ = $\langle -3,9\rangle$

c) 2**u** + (-1)**v** = $2\langle -1,3\rangle + (-1)\langle 4,7\rangle = \langle -2,6\rangle + \langle -4,-7\rangle = \langle -6,-1\rangle$

## Unit Vectors

A vector **u** with length $|\mathbf{u}|$ **= 1** is a **unit vector.** If **v** is not the zero vector $\langle 0,0\rangle$, then the vector $u = \frac{v}{|v|} = \frac{1}{|v|}v$

is a **unit vector in the direction of v.** Unit vectors provide a way to represent the direction of any nonzero vector. Any vector in the direction of **v**, or the opposite direction, is a scalar multiple of this unit vector **u**.

**Example:**

Find a unit vector in the direction of **v** = $\langle -3,2\rangle$, and verify that it has length 1.

$|v| = |\langle -3,2\rangle| = \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$, so

$\frac{v}{|v|} = \frac{1}{\sqrt{13}}\langle -3,2\rangle = \langle \frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}\rangle$

The magnitude of this vector is $\left|\langle\frac{-3}{\sqrt{13}},\frac{2}{\sqrt{13}}\rangle\right| = \sqrt{\left(\frac{-3}{\sqrt{13}}\right)^2+\left(\frac{2}{\sqrt{13}}\right)^2} =$

$$\sqrt{\left(\frac{9}{13}\right)+\left(\frac{4}{13}\right)} = \sqrt{\frac{13}{13}} = 1$$

Standard unit vectors are $i = \langle 1,0\rangle$ *and* $j = \langle 0,1\rangle$

Any vector **v** can be written as an expression in terms of the standard unit vectors: **v**= $\langle a,b\rangle$

$$= \langle a,0\rangle + \langle 0,b\rangle$$

$$= a\langle 1,0\rangle + b\langle 0,1\rangle$$

= *a***i** + *b***j**

Here the vector **v**= $\langle a,b\rangle$ is expressed as the **linear combination** *a***i + ***b***j of** the vectors **i** and **j.** The scalars *a* and *b* are the **horizontal** and **vertical components** of the vector **v.**

**Direction Angles**

A precise way to specify the direction of a vector **v** is to state its **direction angle** (the angle $\theta$ that **v** makes with the positive x-axis). Using trig, see that the horizontal component of v is $|\mathbf{v}|\cos\theta$ and the vertical component is $|\mathbf{v}|\sin\theta$ so, if **v** has direction angle $\theta$, the components of **v** can be computed using the formula

$$v = \langle |v|\cos\theta, |v|\sin\theta\rangle,$$

with linear combination of **V** = ( |**v**| cosθ) ***i*** **+** ( |**v**| sinθ) ***j.***

The unit vector in the direction of v is

$$u = \frac{v}{|v|} = (\cos\theta)i + (\sin\theta)j$$

**Example:**

Find the magnitude and direction angle of each vector.

a) $u = \langle 3, 2\rangle$ b) $v = \langle -2, -5\rangle$

a) $|u| = \sqrt{3^2 + 2^2} = \sqrt{13}$. If $\alpha$ is the direction angle of **u,** then $u = \langle 3, 2\rangle = \langle |u|\cos\alpha, |u|\sin\alpha\rangle$.

$3 = |u|\cos\alpha$ (horizontal component of **u**)

$$3 = \sqrt{3^2 + 2^2}\cos\alpha$$

$$3 = \sqrt{13}\cos\alpha$$

$$\alpha = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right) \approx 33.69^\circ$$

Now try b.

**Velocity** of a moving object is a vector.

(Remember that the bearing of an angle is the angle that the line of travel makes with **DUE NORTH**, **measured clockwise!**)

The magnitude of velocity is **speed.**